Reply to 'Comment on "Timelike flows of energy momentum and particle trajectories for the Klein–Gordon equation"'

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REPLY

Reply to 'Comment on "Timelike flows of energy momentum and particle trajectories for the Klein–Gordon equation"'

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Abstract
We demonstrate that a straightforward limiting process can be used to continue trajectories in spacetime through exceptional regions in the velocity field (identified by Tumulka (2002 Comment on 'Timelike flows of energy–momentum and particle trajectories for the Klein–Gordon equation')) using the formalism we previously proposed (Horton G, Dewdney C and Nesteruk A 2000 *J. Phys. A: Math. Gen.* **33** 7337).

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In our paper [2] we showed how the eigenvectors of the stress energy momentum tensor can be employed to define flow lines in spacetime (and hence particle trajectories which follow the flow lines) for particles obeying the Klein–Gordon equation. The stress energy momentum tensor, $[T_{\mu\nu}]$, for a scalar field, with the usual spacetime metric, will have four eigenvectors, one timelike and three spacelike. In our paper [2] we have identified two eigenvectors explicitly and indicated how to construct the other two. The two eigenvectors $W_\mu^+ = S_\mu + \mathrm{e}^\theta P_\mu$ and $W_\mu^- = S_\mu - \mathrm{e}^{-\theta} P_\mu$ are orthogonal. (Our notation is defined according to $P_\mu = \partial_\mu P$ and $P_{\mu\nu} = \partial_\mu \partial_\nu P$ and similarly for S, $\psi = \mathrm{e}^{(P+\mathrm{i}S)}$, a solution of the Klein–Gordon equation and $\sinh\theta = (\frac{P^\mu P_\mu - S^\mu S_\mu}{2P^\mu S_\mu})$.)

Tumulka [1] raises an interesting point, overlooked in our original paper, concerning the exceptional points where our expressions for W_μ^+ and W_μ^- both become spacelike. For the latter case to occur then, in some reference frame, the time components must disappear and this will only be possible (without contradiction) if S_0 and P_0 vanish.

That such a case can occur at isolated points is illustrated by Tumulka [1] in his specific example but these isolated points do not pose a serious problem. A straightforward computation shows that the points where $S_0 = 0$ form nodal lines as do the points where $P_0 = 0$. The intersections of the two sets of nodal lines are only isolated points at a given time, which then form lines in Minkowski spacetime. So there is no need to ignore such examples. In those cases where the timelike flow lines contain an exceptional point they can be dealt with by using a limiting process from neighbouring ordinary points.

Tumulka, however, goes on to conjecture, but not prove, that since $W^+_\mu W^{+\mu}$ and $W^-_\mu W^{-\mu}$ are continuous functions of P_μ and S_μ that the set of pairs P_μ, S_μ where both W^+_μ and W^-_μ are spacelike is open in the eight-dimensional space of the pairs and has positive measure. He then claims 'that it is reasonable to expect that the *spacetime points* with spacelike W^+_μ and W^-_μ form a set of positive measure' (unlike Tumulka's specific example). The latter claim is crucial since we are interested in trajectories in spacetime but Tumulka provides neither proof nor grounds for his expectation. One needs, therefore, to consider the neighbourhood of a point in Minkowski spacetime with $S_0 = P_0 = 0$ in more detail in order to explore the consequences of continuity.

If at a neighbouring point both W^+_μ and W^-_μ are spacelike then, in some new reference frame, the transformed S'_0 and P'_0 will be zero. A general Lorentz transformation can be composed of a pure spatial rotation followed by a Lorentz boost followed again by another spatial rotation. Neither of the rotations affects the discussion as it does not alter the time components. Using the formula for a Lorentz boost in a general direction the required velocity of the boost is

$$v_k = \widehat{v}_k v \qquad |\underline{v}| = v = \tanh\alpha.$$

(The latin index denotes the space components.) Then considering only the transformation of the timelike components

$$P'_0 = P_0 \cosh\alpha - (\widehat{v}_k P_k)\sinh\alpha = 0$$

where

$$P_0 = P_0(0) + P_{0\mu}(0)\triangle x^\mu$$

and (0) denotes the original point at which $S_0 = P_0 = 0$ and

$$P_k = P_k(0) + P_{k\mu}(0)\triangle x^\mu$$

therefore

$$\cosh\alpha(P_{0\mu}(0)\triangle x^\mu) - \sinh\alpha(\widehat{v}_k P_{k\mu}(0)\triangle x^\mu) + \sinh\alpha(\widehat{v}_k P_k(0)) = 0.$$

One requires

$$\sinh\alpha(\widehat{v}_k P_k(0)) = 0 \tag{1}$$

and

$$(\cosh\alpha(P_{0\mu}(0)) - \sinh\alpha(\widehat{v}_k P_{k\mu}(0)))\triangle x^\mu = 0. \tag{2}$$

An identical calculation will apply to the derivatives of S. Condition (1) can always be fulfilled for P and S by choosing $\widehat{v}_k$ orthogonal to P_k, S_k. Condition (2) could then allow for two three-dimensional regions. (However, the second-order derivatives of P and S will have to be in the appropriate ratios.) The two regions must then intersect to give a common solution giving rise to a one-, two- or three-dimensional region.

Even if these unlikely conditions are all satisfied one never has a four-dimensional spacetime region infected by the disease. One can therefore still take a region outside the exceptional region and use a limiting process on the given 4-velocities $S_\mu + e^\theta P_\mu$ and $S_\mu - e^{-\theta} P_\mu$ to derive trajectories through the exceptional region. So Tumulka was correct to point out that there will be, in general, exceptional regions that require special consideration beyond the prescription given in our paper. However, his arguments do not present an insurmountable difficulty for our approach.

References

[1] Tumulka R 2002 Comment on 'Timelike flows of energy–momentum and particle trajectories for the Klein–Gordon equation' *J. Phys. A: Math. Gen.* **35** 7961

[2] Horton G, Dewdney C and Nesteruk A 2000 *J. Phys. A: Math. Gen.* **33** 7337